MATH 590: QUIZ 5 SOLUTIONS

Name:

1. Let $T : V \to W$ be a linear transformation of vectors spaces with scalars in F. Define: (a) The *kernel* of T and (b) The *image* of T. (2 points)

Solution. The kernel of T is the set of vectors $v \in V$ such that $T(v) = \vec{0}$ and the image of T is the set of vectors $w \in W$ that can be written as $w = T(v)$, for some $v \in V$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (y, x)$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(x, y) = (x + 7, x - y)$. Let $\alpha := \{(-1, 1), (0, 1)\}$, $\beta := \{(1, 0), (1, 1)\}$, and $\gamma := \{(0, -1), (-2, 0)\}$ be bases for $\mathbb{R}^2$. Verify that $[ST]^\gamma_\alpha = [S]^\gamma_\beta \cdot [T]^\beta_\alpha$. (4 points)

Solution. We have $ST(-1, 1) = S(1, -1) = (8, 2) = -2(0, -1) + -4(-2, 0)$ and $ST(0, 1) = S(1, 0) = (8, 1) = -1(0, -1) + -4(-2, 0)$. Thus, $[ST]^\gamma_\alpha = \begin{pmatrix} -2 & -1 \\ -4 & -4 \end{pmatrix}$. On the other hand,

$T(-1, 1) = (1, -1) = 2(1, 0) + -1(1, 1)$ and $T(0, 1) = (1, 0) = 1(1, 0) + 0(1, 1)$. Thus, $[T]^\beta_\alpha = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

$S(1, 0) = (8, 1) = -1(0, -1) + -4(-2, 0)$ and $S(1, 1) = (8, 0) = 0(0, -1) + -4(-2, 0)$. Thus, $]S]^\gamma_\beta = \begin{pmatrix} -1 & 0 \\ -4 & -4 \end{pmatrix}$.

Therefore,

$$[S]^\gamma_\beta \cdot [T]^\beta_\alpha = \begin{pmatrix} -1 & 0 \\ -4 & -4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -4 & -4 \end{pmatrix} = [ST]^\gamma_\alpha.$$

3. $A = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$, and write C_1, C_2, C_3 for the columns of A. Verify that $A \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = uC_1 + vC_2 + wC_3$. (4 points)

Solution. On the one hand, $A \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} ua + vd + wg \\ ub + ve + wh \\ uc + vf + wi \end{pmatrix}$. While on the other hand,

$$uC_1 + vC_2 + wC_3 = u \begin{pmatrix} a \\ b \\ c \end{pmatrix} + v \begin{pmatrix} d \\ e \\ f \end{pmatrix} + w \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \begin{pmatrix} ua + vd + wg \\ ub + ve + wh \\ uc + vf + wi \end{pmatrix}$$

which gives what we want.